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Spanning trees on two-dimensional lattices with more than one type of vertex

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Abstract

For a two-dimensional lattice Λ with n vertices, the number of spanning trees $N_{\mathrm{ST}}(\Lambda)$ grows asymptotically as $\exp(nz_\Lambda)$ in the thermodynamic limit. We present an exact integral expression and a numerical value for the entropy (asymptotic growth constant) z_Λ for spanning trees on 19 two-dimensional lattices with more than one type of vertex given in O'Keeffe and Hyde (1980 *Philos. Trans. R. Soc.* A **295** 553). Especially, an exact closed-form expression for the entropy is derived for net 14, and the entropies of net 27 and the triangle lattice have the simple relation $z_{27} = (z_{\mathrm{tri}} + \ln 4)/4$. Some integral identities are also obtained.

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1. Introduction

The enumeration of the number of spanning trees $N_{\mathrm{ST}}(G)$ on the graph G was first considered by Kirchhoff in the analysis of electric circuits [2]. It is a problem of fundamental interest in mathematics [3–6] and physics [7, 8]. It is well known that there is a bijection between close-packed dimer coverings with spanning tree configurations on two related lattices [9]. The number of spanning trees is closely related to the partition function of the q-state Potts model in statistical mechanics [10, 11] and loop-erased self-avoiding walk [12, 13]. The spanning trees problem has various applications in many areas; see, for example, [14] and references therein. There are several ways to calculate $N_{\mathrm{ST}}(G)$, including as a determinant of the Laplacian matrix of G and as a special case of the Tutte polynomial of G [3]. Some recent studies on the enumeration of spanning trees and the calculation of their entropies, or asymptotic growth constants, on regular lattices were carried out in [15–18]. In this paper we shall present exact integrals for the entropy for spanning trees on several two-dimensional lattices given in [1], and obtain two integral identities with different choice of unit cells as in [18].

2. Background and method

We briefly recall some definitions and background on spanning trees and the calculation method that we use [3, 19]. Let $G = (V, E)$ denote a connected graph (without loops) with vertex (site) set V, edge (bond) set E, and an incidence relation. We will only consider simple graphs without multiple edges. Let $n = v(G) = |V|$ be the number of vertices and $e(G) = |E|$ the number of edges in G. A spanning subgraph G' is a subgraph of G with $v(G') = |V|$, and a tree is a connected graph with no circuits. It follows that a spanning tree is a spanning subgraph of G that is a tree and hence $e(G') = n - 1$. The degree or coordination number k_i of a vertex $v_i \in V$ is the number of edges attached to it. A k-regular graph is a graph with the property that each of its vertices has the same degree k. Two vertices are adjacent if they are connected by an edge in E. The adjacency matrix $A(G)$ of G is the $n \times n$ matrix with elements $A(G)_{ij} = 1$ if v_i and v_j are adjacent and zero otherwise. The Laplacian matrix $Q(G)$ is the $n \times n$ matrix with the element $Q(G)_{ij} = k_i \delta_{ij} - A(G)_{ij}$. One of the eigenvalues of $Q(G)$ is always zero; let us denote the rest as $\lambda(G)_i$, $1 \leqslant i \leqslant n - 1$. A basic theorem is that $N_{\mathrm{ST}}(G) = (1/n) \prod_{i=1}^{n-1} \lambda(G)_i$ [3]. A lattice Λ can be regarded as a graph with a repeated subgraph. For a d-dimensional lattice with $d \geqslant 2$ in the thermodynamic limit, $N_{\mathrm{ST}}(\Lambda)$ grows exponentially with $n(\Lambda)$ as $n(\Lambda) \to \infty$ such that the number of repeated subgraphs is infinite in each direction; that is, there exists a constant z_Λ such that $N_{\mathrm{ST}}(\Lambda) \sim \exp(n(\Lambda) z_\Lambda)$ as $n(\Lambda) \to \infty$. The constant describing this exponential growth is the entropy given by [5, 6]

$$z_\Lambda = \lim_{n(\Lambda)\to\infty} \frac{1}{n(\Lambda)} \ln[N_{\mathrm{ST}}(\Lambda)], \tag{1}$$

where Λ, when used as a subscript in this manner, implicitly refers to the thermodynamic limit of the lattice Λ.

A regular d-dimensional lattice is comprised of repeated unit cells, each containing ν vertices. Once a specific vertex labeling inside a unit cell is chosen and the coordinates for the unit cells are specified as illustrated in the following figures, define $a(\tilde{n}, \tilde{n}')$ as the $\nu \times \nu$ matrix describing the adjacency of the vertices of the unit cells $\tilde{n}$ and $\tilde{n}'$, the elements of which are given by $a(\tilde{n}, \tilde{n}')_{ij} = 1$ if $v_i \in \tilde{n}$ is adjacent to $v_j \in \tilde{n}'$ and zero otherwise. Although the number of spanning trees $N_{\mathrm{ST}}(\Lambda)$ depends on the boundary conditions imposed as shown in [15], the entropy z_Λ is not sensitive to them. For simplicity, let us consider a given lattice with periodic boundary conditions. Using the resultant translational symmetry for spanning trees, we have $a(\tilde{n}, \tilde{n}') = a(\tilde{n} - \tilde{n}')$, and we can therefore write $a(\tilde{n}) = a(\tilde{n}_1, \ldots, \tilde{n}_d)$ for a d-dimensional lattice. Generalizing the method derived in [16] for lattices which are not k-regular, $N_{\mathrm{ST}}(\Lambda)$ and z_Λ can be calculated in terms of a matrix M_Λ, which is determined by these $a(\tilde{n})$, defined as

$$M_\Lambda(\theta_1, \ldots, \theta_d) = M'_\Lambda - \sum_{\tilde{n}} a(\tilde{n})\, \mathrm{e}^{\mathrm{i}\tilde{n}\cdot\Theta}, \tag{2}$$

where M'_Λ is the diagonal matrix whose diagonal elements are the degrees k_i of the vertices in the unit cell and Θ stands for the d-dimensional vector $(\theta_1, \ldots, \theta_d)$. Then [5, 16]

$$z_\Lambda = \frac{1}{\nu} \int_{-\pi}^{\pi} \left[\prod_{j=1}^{d} \frac{\mathrm{d}\theta_j}{2\pi} \right] \ln[D_\Lambda(\theta_1, \ldots, \theta_d)], \tag{3}$$

where $D_\Lambda(\theta_1, \ldots, \theta_d) = \det(M_\Lambda(\theta_1, \ldots, \theta_d))$ is the determinant of the matrix M_Λ, invariant with respect to the order of vertex labeling. Note that the calculation is not sensitive to the choice of the directions θ_j, $1 \leqslant j \leqslant d$, too, that will be illustrated. We shall only consider two-dimensional lattices with $d = 2$ throughout this paper.

It is well known that there are only three uniform tilings of the plane by using one type of regular polygon in which all vertices are equivalent, or three regular tessellations, namely, the square, triangular and honeycomb lattices. If one allows more than one kind of regular polygons and still requests that all vertices are equivalent, there are eight more lattices or semi-regular tessellations. These are altogether 11 Archimedean lattices which are all k-regular [20]. If the restriction that all vertices are equivalent is released, an infinite number of tessellations are possible, even with just two types of regular polygons, not to mention if non-regular polygons are allowed. Even though mathematically it is not possible to cover the plane if regular pentagons or heptagons should be present, certain arrangements of atoms involving irregular polygons, including pentagons or heptagons, do occur in real-world alloys and inorganic crystals. In [1] various common tessellations (including 11 Archimedean lattices), denoted as nets, and their occurrences were given.

For a lattice Λ which is not k-regular, it is convenient to introduce an effective coordination number κ_Λ, defined as the average number of edges per vertex,

$$\kappa_\Lambda = \lim_{n(\Lambda)\to\infty} \frac{2e(\Lambda)}{n(\Lambda)}. \tag{4}$$

For a k-regular lattice, $\kappa = k$. Furthermore, we know that the number of spanning trees is the same for a planar graph G and its dual G^*, and the number of the vertices of G^* is given by the Euler relation $v(G^*) = e(G) - n + 1$. It follows that the entropies of G and G^* satisfy the relation [16, 18]

$$z_{G^*} = \frac{z_G}{\kappa/2 - 1}. \tag{5}$$

For a k-regular graph G_k, a general upper bound for the entropy is $z_{G_k} \leqslant \ln k$ [21]. A stronger upper bound for G_k with $k \geqslant 3$ was derived in [22, 23] that

$$N_{\mathrm{ST}}(G_k) \leqslant \left(\frac{2\ln n}{nk\ln k}\right)(b_k)^n, \tag{6}$$

where

$$b_k = \frac{(k-1)^{k-1}}{[k(k-2)]^{\frac{k}{2}-1}}. \tag{7}$$

By equation (1), this then yields [16]

$$z_{G_k} \leqslant \ln(b_k). \tag{8}$$

3. Entropies

The entropies z_Λ for 11 Archimedean lattices, denoted as nets 1–11 in [1], have been considered by several authors [7, 8, 16–18]. While the relation $z_{\mathrm{hc}} = z_{\mathrm{tri}}/2$ for the honeycomb and triangular lattices is easy to understand due to the duality [16] (cf equation (5)), it is non-trivial to have the relations $z_{\mathrm{kag}} = (z_{\mathrm{tri}} + \ln 6)/3$ for the Kagomé (equivalently (3.6.3.6)) lattice and $z_{(3.12.12)} = (z_{\mathrm{tri}} + \ln(15))/6$ for the (3.12.12) lattice given in [16]. Our main purpose is to calculate the entropies for other common two-dimensional lattices where more than one type of vertex occurs. Following the denotation given in [1], we shall quote them as nets 12–27, where nets 12–17 are tessellations with two or three regular polygons (including triangle, square or hexagon), while nets 18–27 are tessellations containing pentagons, heptagons or enneagons. In addition, [1] mentions the B net of $YCrB_4$, the B net of Y_2 LnB_6 and a net with $5^2.8$ and 5.8^2 vertices. Let us denote them as nets 28, 29 and 30, respectively. The figures of these nets are referred to those in [1], and the unit cells chosen for the calculation are shown in figures 1–3. Note that the number of spanning trees is not affected by whether or not the polygons are regular, so the unit cells can be deformed from those in [1]. In addition to the numerical values of the entropies, we shall derive an exact closed-form expression for z_{14} and the relation $z_{27} = (z_{\mathrm{tri}} + \ln 4)/4$. Two integral identities will be given by choosing different unit cells in the calculation for nets 24 and 30. We will use the shorthand notations

$$\alpha = \mathrm{e}^{\mathrm{i}\theta_1} \qquad \text{and} \qquad \beta = \mathrm{e}^{\mathrm{i}\theta_2} \tag{9}$$

for the elements of the matrix M_Λ when the matrix is large.

3.1. Nets with regular polygons

In this subsection, we consider important nets in which only regular polygons (including triangle, square or hexagon) occur with more than one type of vertex.

3.1.1. Net 12. Net 12 is the combination of $3^2.4.3.4$ and $3^3.4^2$ vertices. A primitive unit cell contains 12 vertices $\nu_{12} = 12$, and the coordination number is $k_{12} = 5$. By the choice of the unit cell and vertex labeling shown in figure 1(*a*), we have

$$M_{12}(\theta_1, \theta_2) = \begin{pmatrix} 5 & -1 & 0 & -1 & 0 & -\alpha & 0 & 0 & 0 & -1 & -1 & 0 \\ -1 & 5 & -1 & 0 & -\alpha & -\alpha & 0 & -\alpha\beta & 0 & 0 & 0 & 0 \\ 0 & -1 & 5 & -1 & 0 & 0 & -\alpha\beta & 0 & -\beta & -\beta & 0 & 0 \\ -1 & 0 & -1 & 5 & -1 & 0 & 0 & 0 & -\beta & 0 & -1 & 0 \\ 0 & -\frac{1}{\alpha} & 0 & -1 & 5 & -1 & 0 & -\beta & 0 & 0 & -1 & 0 \\ -\frac{1}{\alpha} & -\frac{1}{\alpha} & 0 & 0 & -1 & 5 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -\frac{1}{\alpha\beta} & 0 & 0 & -1 & 5 & -1 & 0 & -\frac{1}{\alpha} & 0 & -1 \\ 0 & -\frac{1}{\alpha\beta} & 0 & 0 & -\frac{1}{\beta} & 0 & -1 & 5 & -1 & 0 & 0 & -1 \\ 0 & 0 & -\frac{1}{\beta} & -\frac{1}{\beta} & 0 & 0 & 0 & -1 & 5 & -1 & 0 & -1 \\ -1 & 0 & -\frac{1}{\beta} & 0 & 0 & 0 & -\alpha & 0 & -1 & 5 & -1 & 0 \\ -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 5 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & -1 & 5 \end{pmatrix}. \tag{10}$$

The determinant can be calculated to be

$$\begin{aligned} D_{12}(\theta_1, \theta_2) = 16\{&1650732 - 680016(\cos\theta_1 + \cos\theta_2) + 10151(\cos^2\theta_1 + \cos^2\theta_2) \\ &- 300022\cos\theta_1\cos\theta_2 - (\cos^3\theta_1 + \cos^3\theta_2) \\ &- 5567\cos\theta_1\cos\theta_2(\cos\theta_1 + \cos\theta_2) + 158\cos^2\theta_1\cos^2\theta_2 \\ &- \cos\theta_1\cos\theta_2(\cos^2\theta_1 + \cos^2\theta_2)\}, \end{aligned} \tag{11}$$

such that the numerical evaluation (using, e.g., Maple) gives

$$z_{12} = \frac{1}{12}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_2}{2\pi}\ln[D_{12}(\theta_1, \theta_2)] = 1.409737903756929\ldots. \tag{12}$$

3.1.2. Net 13. Net 13 is obtained by the intergrowth of $3^2.4.3.4$ and $3^3.4^2$ vertices. It can be constructed by starting with the square lattice and adding appropriate diagonal edges as shown in figure 1(*b*), such that the coordination number is $k_{13} = 5$. Taking three contiguous

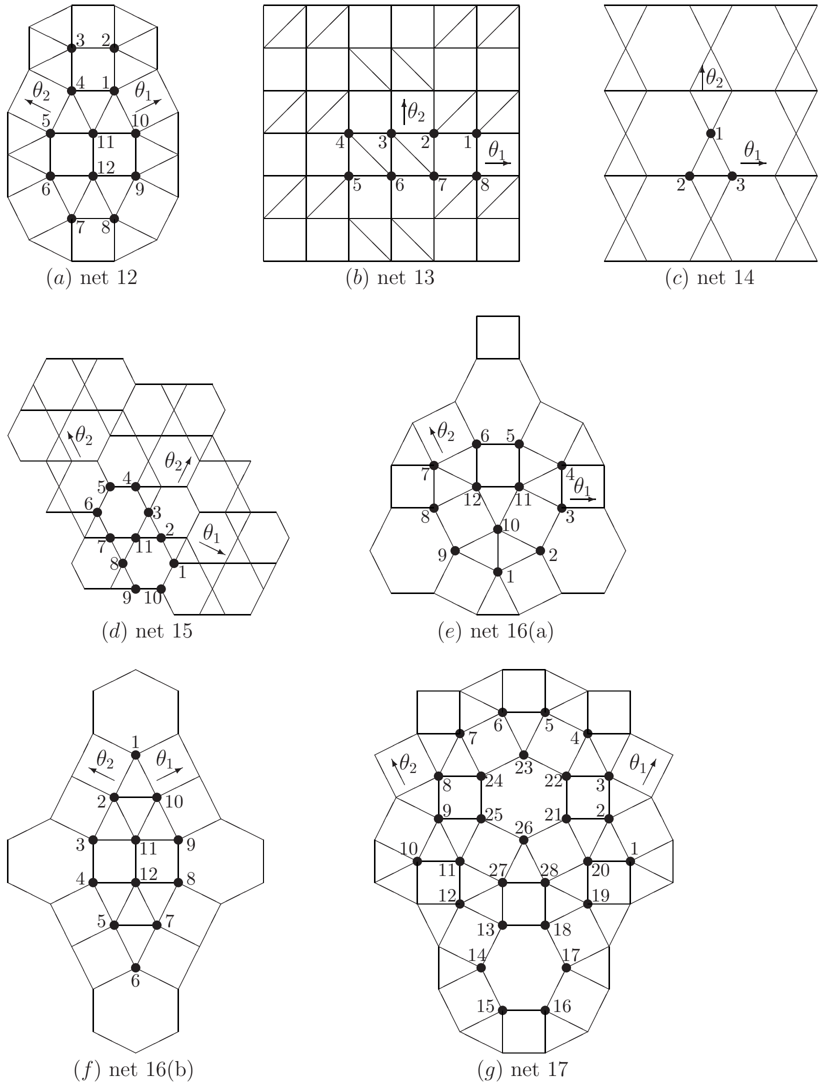


Figure 1. Unit cells for nets 12–17. Vertices within a unit cell are labeled. Directions θ_1 and θ_2 are sketched. There are two choices of θ_2 for net 15.

squares where two of them contain diagonal edges as a unit cell with $\nu_{13} = 8$, we have

$$M_{13}(\theta_1,\theta_2) = \begin{pmatrix} 5 & -1 & 0 & -\alpha & -\alpha\beta & 0 & 0 & -1-\beta \\ -1 & 5 & -1 & 0 & 0 & 0 & -1-\beta & -\beta \\ 0 & -1 & 5 & -1 & 0 & -1-\beta & -1 & 0 \\ -\frac{1}{\alpha} & 0 & -1 & 5 & -1-\beta & -1 & 0 & 0 \\ -\frac{1}{\alpha\beta} & 0 & 0 & -1-\frac{1}{\beta} & 5 & -1 & 0 & -\frac{1}{\alpha} \\ 0 & 0 & -1-\frac{1}{\beta} & -1 & -1 & 5 & -1 & 0 \\ 0 & -1-\frac{1}{\beta} & -1 & 0 & 0 & -1 & 5 & -1 \\ -1-\frac{1}{\beta} & -\frac{1}{\beta} & 0 & 0 & -\alpha & 0 & -1 & 5 \end{pmatrix}. \tag{13}$$

The determinant can be calculated to be

$$D_{13}(\theta_1,\theta_2) = 4\{25560 - 2328\cos\theta_1 - 25872\cos\theta_2 + \cos^2\theta_1 + 5397\cos^2\theta_2 - 2158\cos\theta_1\cos\theta_2 - 288\cos^3\theta_2 - 312\cos\theta_1\cos^2\theta_2 + 4\cos^4\theta_2 - 4\cos\theta_1\cos^3\theta_2\}, \tag{14}$$

such that

$$z_{13} = \frac{1}{8}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\ln[D_{13}(\theta_1,\theta_2)] = 1.409\,133\,286\,424\,679\ldots. \tag{15}$$

The numerical values of the entropies for the variants of net 13 shown in figure 15 of [1] are close to this, and are not given here to save space.

3.1.3. Net 14. Net 14 is a simple combination of hexagons and triangles as shown in figure 1(*c*), such that the coordination number is $k_{14} = 4$. Taking a triangle as a unit cell with $\nu_{14} = 3$, we have

$$M_{14}(\theta_1,\theta_2) = \begin{pmatrix} 4 & -1-e^{i\theta_2} & -1-e^{i\theta_2} \\ -1-e^{-i\theta_2} & 4 & -1-e^{-i\theta_1} \\ -1-e^{-i\theta_2} & -1-e^{i\theta_1} & 4 \end{pmatrix}. \tag{16}$$

The determinant can be calculated to be

$$D_{14}(\theta_1,\theta_2) = 4(9 - 3\cos\theta_1 - 5\cos\theta_2 - \cos\theta_1\cos\theta_2), \tag{17}$$

such that

$$\begin{aligned} z_{14} &= \frac{1}{3}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\ln[D_{14}(\theta_1,\theta_2)] \\ &= \frac{\ln 4}{3} + \frac{1}{3}\int_0^{\pi}\frac{d\theta_1}{\pi}\ln\left[\frac{9 - 3\cos\theta_1 + \sqrt{8(1-\cos\theta_1)(7-\cos\theta_1)}}{2}\right] \\ &= \frac{\ln 2}{3} + \frac{2}{3}\int_0^{\pi}\frac{d\theta_1}{\pi}\ln\left(2\sin\frac{\theta_1}{2} + \sqrt{6 + 2\sin^2\frac{\theta_1}{2}}\right). \end{aligned} \tag{18}$$

An exact closed-form expression for this integral can be derived as follows. After changing the variable $\theta_1 = 2\theta$, we have

$$\begin{aligned} z_{14} &= \ln 2 + \frac{4}{3\pi}\int_0^{\pi/2} d\theta\,\ln\left(\sin\theta + \sqrt{\frac{3}{2} + \frac{1}{2}\sin^2\theta}\right) \\ &= \ln 2 + \frac{4}{3}I\left(\frac{3}{2},\frac{1}{2}\right), \end{aligned} \tag{19}$$

where

$$I(a,b) = \frac{1}{\pi}\int_0^{\pi/2} d\theta\,\ln(\sin\theta + \sqrt{a + b\sin^2\theta}) \tag{20}$$

and we consider $0 \leqslant b < 1 \leqslant a$. For $b = 0$, it can be shown that

$$\begin{aligned} I(a,0) &= \frac{1}{\pi}\int_0^{\pi/2} d\theta\,\ln(\sin\theta + \sqrt{a}) \\ &= -\frac{1}{2}\ln[2(\sqrt{a} + \sqrt{a-1})] + \frac{2}{\pi}\mathrm{Ti}_2(\sqrt{a} + \sqrt{a-1}), \end{aligned} \tag{21}$$

where $\mathrm{Ti}_2(x)$ is the inverse tangent integral [24],

$$\mathrm{Ti}_2(x) = \int_0^x \frac{\tan^{-1} t}{t}\,\mathrm{d}t = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)^2} = \frac{1}{2\mathrm{i}}[\mathrm{Li}_2(\mathrm{i}x) - \mathrm{Li}_2(-\mathrm{i}x)]. \tag{22}$$

Here the dilogarithm $\mathrm{Li}_2(z)$ is defined by

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} = \int_z^0 \frac{\ln(1-t)}{t}\,\mathrm{d}t. \tag{23}$$

Taking the derivative with respect to b then performing the integral over θ in equation (20), we get

$$\frac{\mathrm{d}}{\mathrm{d}b} I(a,b) = \frac{1}{2\pi(1-b)} \left(\frac{1}{\sqrt{b}} \tan^{-1} \sqrt{\frac{b}{a}} + \sqrt{\frac{a}{a+b-1}} \tan^{-1} \sqrt{a+b-1} \right) - \frac{1}{4(1-b)}. \tag{24}$$

It can be integrated to give

$$\begin{aligned}
I(a,b) &= I(a,0) + \int_0^b I'(a,x)\,\mathrm{d}x \\
&= \frac{1}{2} \ln\left[\frac{\sqrt{1-b}}{2(\sqrt{a}+\sqrt{a-1})} \right] + \frac{2}{\pi} \mathrm{Ti}_2(\sqrt{a}+\sqrt{a-1}) \\
&\quad + \frac{1}{\pi} \left(\tanh^{-1} \sqrt{1-\frac{1-b}{a}} - \tanh^{-1} \sqrt{1-\frac{1}{a}} \right) \tan^{-1} \sqrt{a} \\
&\quad + \frac{1}{\pi} (\tanh^{-1} \sqrt{b}) \left(\tan^{-1} \frac{1}{\sqrt{a}} \right) + \frac{1}{2\pi} \sum_{k=1}^{\infty} \left[\left(\frac{\sqrt{a-1}-\sqrt{a}}{\sqrt{a-1}+\sqrt{a}} \right)^k \right. \\
&\quad \left. - \left(\frac{\sqrt{a+b-1}-\sqrt{a}}{\sqrt{a+b-1}+\sqrt{a}} \right)^k + \left(\frac{1-\sqrt{b}}{1+\sqrt{b}} \right)^k - 1 \right] \frac{\sin(k\phi)}{k^2} \\
&= \frac{1}{4} \ln\left[\frac{(1-b)(1+\sqrt{b})}{4(2a-1+2\sqrt{a(a-1)})(1-\sqrt{b})} \right] + \frac{2}{\pi} \mathrm{Ti}_2(\sqrt{a}+\sqrt{a-1}) \\
&\quad + \frac{1}{\pi} \left(\tanh^{-1} \sqrt{1-\frac{1-b}{a}} - \tanh^{-1} \sqrt{1-\frac{1}{a}} - \tanh^{-1} \sqrt{b} \right) \tan^{-1} \sqrt{a} \\
&\quad + \frac{1}{4\pi\mathrm{i}} \left[\mathrm{Li}_2\left(\frac{\sqrt{a-1}-\sqrt{a}}{\sqrt{a-1}+\sqrt{a}} \mathrm{e}^{\mathrm{i}\phi} \right) - \mathrm{Li}_2\left(\frac{\sqrt{a-1}-\sqrt{a}}{\sqrt{a-1}+\sqrt{a}} \mathrm{e}^{-\mathrm{i}\phi} \right) \right. \\
&\quad - \mathrm{Li}_2\left(\frac{\sqrt{a+b-1}-\sqrt{a}}{\sqrt{a+b-1}+\sqrt{a}} \mathrm{e}^{\mathrm{i}\phi} \right) + \mathrm{Li}_2\left(\frac{\sqrt{a+b-1}-\sqrt{a}}{\sqrt{a+b-1}+\sqrt{a}} \mathrm{e}^{-\mathrm{i}\phi} \right) \\
&\quad \left. + \mathrm{Li}_2\left(\frac{1-\sqrt{b}}{1+\sqrt{b}} \mathrm{e}^{\mathrm{i}\phi} \right) - \mathrm{Li}_2\left(\frac{1-\sqrt{b}}{1+\sqrt{b}} \mathrm{e}^{-\mathrm{i}\phi} \right) - \mathrm{Li}_2(\mathrm{e}^{\mathrm{i}\phi}) + \mathrm{Li}_2(\mathrm{e}^{-\mathrm{i}\phi}) \right],
\end{aligned} \tag{25}$$

where $\phi = \tan^{-1}[2\sqrt{a}/(1-a)]$ and $\pi/2 \leqslant \phi < \pi$. When b is set to zero, it is clear that equation (25) reduces to equation (21). We note that when a is set to 1, the expression for

$I(a = 1, b)$ can be simplified, using the identity that $\mathrm{Ti}_2(1)$ is equal to the Catalan constant $C = \sum_{n=0}^{\infty}(-1)^n(2n+1)^{-2} = 0.915\,965\,594\,177\,2190\ldots$, as

$$I(1,b) = \frac{1}{4}\ln\left(\frac{1-b}{4}\right) + \frac{C}{\pi} + \frac{1}{2}\tanh^{-1}\sqrt{b} + \frac{1}{\pi}\mathrm{Ti}_2\left(\frac{1-\sqrt{b}}{1+\sqrt{b}}\right), \tag{26}$$

which is equivalent to equation (28) of [17]. Evaluating $I(a,b)$ in equation (25) at $a = 3/2, b = 1/2$ and substituting into equation (19), we obtain the exact closed-form expression

$$\begin{aligned} z_{14} = {} & \frac{1}{3}\ln\left(\frac{3+2\sqrt{2}}{2+\sqrt{3}}\right) + \frac{4}{3\pi}\left(\tan^{-1}\sqrt{\frac{3}{2}}\right)\left(\tanh^{-1}\frac{4\sqrt{3}-\sqrt{6}-7}{\sqrt{3}-2\sqrt{6}+7\sqrt{2}}\right) \\ & + \frac{8}{3\pi}\mathrm{Ti}_2\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right) + \frac{1}{3\pi\mathrm{i}}[\mathrm{Li}_2((\sqrt{3}-2)\,\mathrm{e}^{\mathrm{i}\phi_0}) - \mathrm{Li}_2((\sqrt{3}-2)\,\mathrm{e}^{-\mathrm{i}\phi_0}) \\ & - \mathrm{Li}_2((2\sqrt{6}-5)\mathrm{e}^{\mathrm{i}\phi_0}) + \mathrm{Li}_2((2\sqrt{6}-5)\,\mathrm{e}^{-\mathrm{i}\phi_0}) + \mathrm{Li}_2((3-2\sqrt{2})\,\mathrm{e}^{\mathrm{i}\phi_0}) \\ & - \mathrm{Li}_2((3-2\sqrt{2})\,\mathrm{e}^{-\mathrm{i}\phi_0}) - \mathrm{Li}_2(\mathrm{e}^{\mathrm{i}\phi_0}) + \mathrm{Li}_2(\mathrm{e}^{-\mathrm{i}\phi_0})] \\ = {} & 1.127\,778\,363\,805\,542\ldots, \end{aligned} \tag{27}$$

where $\phi_0 = \tan^{-1}(-2\sqrt{6}) = 1.772\,154\,247\,585\,227\ldots$.

3.1.4. Net 15. Similar to the Kagomé lattice, (3.6.3.6), net 15 is a simple combination of hexagons and triangles as shown in figure 1(*d*). A primitive unit cell contains 11 vertices $\nu_{15} = 11$. There are two 6^3, four $3^2.6^2$ and five 3.6.3.6 vertices in each unit cell, so that the effective coordination number is $\kappa_{15} = 42/11$. Referring to figure 1(*d*), if one takes the direction to the lower right as θ_1 and the direction to the upper right as θ_2, then

$$M_{15}(\theta_1,\theta_2) = \begin{pmatrix} 4 & -1 & 0 & 0 & 0 & -\alpha & -\alpha & 0 & 0 & -1 & 0 \\ -1 & 4 & -1 & 0 & 0 & -\alpha & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & -\beta & 0 & -1 \\ 0 & 0 & -1 & 4 & -1 & 0 & 0 & -\beta & -\beta & 0 & 0 \\ 0 & 0 & 0 & -1 & 3 & -1 & 0 & 0 & 0 & -\frac{\beta}{\alpha} & 0 \\ -\frac{1}{\alpha} & -\frac{1}{\alpha} & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 \\ -\frac{1}{\alpha} & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -\frac{1}{\beta} & 0 & 0 & -1 & 4 & -1 & 0 & -1 \\ 0 & 0 & -\frac{1}{\beta} & -\frac{1}{\beta} & 0 & 0 & 0 & -1 & 4 & -1 & 0 \\ -1 & 0 & 0 & 0 & -\frac{\alpha}{\beta} & 0 & 0 & 0 & -1 & 3 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 4 \end{pmatrix}, \tag{28}$$

with determinant

$$\begin{aligned} D_{15}(\theta_1,\theta_2) = 4\{ & 399\,10 - 169\,05(\cos\theta_1 + \cos\theta_2) - 1270\cos(\theta_1-\theta_2) \\ & + 356(\cos^2\theta_1 + \cos^2\theta_2) - 5352\cos\theta_1\cos\theta_2 \\ & - 25(\cos\theta_1+\cos\theta_2)\cos(\theta_1-\theta_2) - 70\cos\theta_1\cos\theta_2(\cos\theta_1+\cos\theta_2)\}. \end{aligned} \tag{29}$$

However, if one still takes the direction to the lower right as θ_1 but the direction to the upper left as θ_2, then

$$\bar{M}_{15}(\theta_1,\theta_2) = \begin{pmatrix} 4 & -1 & 0 & 0 & 0 & -\alpha & -\alpha & 0 & 0 & -1 & 0 \\ -1 & 4 & -1 & 0 & 0 & -\alpha & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & -\alpha\beta & 0 & -1 \\ 0 & 0 & -1 & 4 & -1 & 0 & 0 & -\alpha\beta & -\alpha\beta & 0 & 0 \\ 0 & 0 & 0 & -1 & 3 & -1 & 0 & 0 & 0 & -\beta & 0 \\ -\frac{1}{\alpha} & -\frac{1}{\alpha} & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 \\ -\frac{1}{\alpha} & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -\frac{1}{\alpha\beta} & 0 & 0 & -1 & 4 & -1 & 0 & -1 \\ 0 & 0 & -\frac{1}{\alpha\beta} & -\frac{1}{\alpha\beta} & 0 & 0 & 0 & -1 & 4 & -1 & 0 \\ -1 & 0 & 0 & 0 & -\frac{1}{\beta} & 0 & 0 & 0 & -1 & 3 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 4 \end{pmatrix}, \tag{30}$$

with determinant

$$\begin{aligned} \bar{D}_{15}(\theta_1,\theta_2) = 4\{&402\,66 - 169\,75\cos\theta_1 - 1270\cos\theta_2 - 169\,05\cos(\theta_1+\theta_2) - 356\cos^2\theta_2 \\ &- 25\cos\theta_1\cos\theta_2 - 5352\cos\theta_1\cos(\theta_1+\theta_2) - 25\cos\theta_2\cos(\theta_1+\theta_2) \\ &+ 70\cos^3\theta_1 + 70\cos\theta_1\cos^2\theta_2 - 70\cos^2\theta_1\cos(\theta_1+\theta_2) \\ &+ 712\cos\theta_1\cos\theta_2\cos(\theta_1+\theta_2) - 140\cos^2\theta_1\cos\theta_2\cos(\theta_1+\theta_2)\}. \end{aligned} \tag{31}$$

Although $\bar{D}_{15}(\theta_1,\theta_2)$ looks distinct from $D_{15}(\theta_1,\theta_2)$, $\bar{D}_{15}(\theta_1,\theta_2-\theta_1)$ is equivalent to $D_{15}(\theta_1,\theta_2)$. Therefore, both determinants give the same entropy

$$\begin{aligned} z_{15} &= \frac{1}{11}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\ln[D_{15}(\theta_1,\theta_2)] \\ &= \frac{1}{11}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\ln[\bar{D}_{15}(\theta_1,\theta_2)] = 1.073\,270\,254\,423\,056\ldots. \end{aligned} \tag{32}$$

This is an illustration that the choice of directions θ_2 does not affect the result.

3.1.5. Net 16. Net 16 is a combination of triangles, squares and hexagons. There are two kinds of nets 16, namely net 16(a) and net 16(b). Their unit cells are shown in figures 1(*e*) and (*f*), respectively, and both of them contain 12 vertices $\nu_{16(a)} = \nu_{16(b)} = 12$. For net 16(a), there are six $3^2.4.3.4$ and six 3.4.6.4 vertices in each unit cell, so that the effective coordination number is $\kappa_{16(a)} = 9/2$. We have

$$M_{16(a)}(\theta_1,\theta_2) = \begin{pmatrix} 5 & -1 & 0 & -\frac{1}{\alpha\beta} & 0 & 0 & -\frac{1}{\beta} & 0 & -1 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 & -\frac{1}{\beta} & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & 0 & 0 & -\alpha & 0 & 0 & -1 & 0 \\ -\alpha\beta & 0 & -1 & 5 & -1 & 0 & -\alpha & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -\alpha\beta & 0 & -1 & 0 \\ 0 & -\beta & 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & -1 \\ -\beta & 0 & 0 & -\frac{1}{\alpha} & 0 & -1 & 5 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -\frac{1}{\alpha} & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & -\frac{1}{\alpha\beta} & 0 & 0 & -1 & 4 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 5 & -1 & -1 \\ 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & 5 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & -1 & 5 \end{pmatrix}. \tag{33}$$

The determinant can be calculated to be

$$
\begin{aligned}
D_{16(a)}(\theta_1,\theta_2) = 112\{&49746 - 15900(\cos\theta_1+\cos\theta_2) - 15899\cos(\theta_1+\theta_2) \\
&- 749\cos\theta_1\cos\theta_2 - 749(\cos\theta_1+\cos\theta_2)\cos(\theta_1+\theta_2) \\
&+ (\cos^3\theta_1+\cos^3\theta_2) - (\cos^2\theta_1+\cos^2\theta_2)\cos(\theta_1+\theta_2) \\
&+ 204\cos\theta_1\cos\theta_2\cos(\theta_1+\theta_2) \\
&- 2\cos\theta_1\cos\theta_2(\cos\theta_1+\cos\theta_2)\cos(\theta_1+\theta_2)\},
\end{aligned}
\tag{34}
$$

such that

$$
z_{16(a)} = \frac{1}{12}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\ln[D_{16(a)}(\theta_1,\theta_2)] = 1.280\,287\,248\,642\,483\ldots. \tag{35}
$$

For net 16(b), there are six $3^3.4^2$ and six 3.4.6.4 vertices in each unit cell, so that the effective coordination number is also $\kappa_{16(b)} = 9/2$. We have

$$
M_{16(b)}(\theta_1,\theta_2) = \begin{pmatrix}
4 & -1 & 0 & -\alpha & 0 & 0 & 0 & -\beta & 0 & -1 & 0 & 0 \\
-1 & 5 & -1 & 0 & 0 & 0 & -\beta & 0 & 0 & -1 & -1 & 0 \\
0 & -1 & 4 & -1 & 0 & -\beta & 0 & 0 & 0 & 0 & -1 & 0 \\
-\frac{1}{\alpha} & 0 & -1 & 4 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 5 & -1 & -1 & 0 & 0 & -\frac{1}{\alpha} & 0 & -1 \\
0 & 0 & -\frac{1}{\beta} & 0 & -1 & 4 & -1 & 0 & -\frac{1}{\alpha} & 0 & 0 & 0 \\
0 & -\frac{1}{\beta} & 0 & 0 & -1 & -1 & 5 & -1 & 0 & 0 & 0 & -1 \\
-\frac{1}{\beta} & 0 & 0 & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -\alpha & 0 & -1 & 4 & -1 & -1 & 0 \\
-1 & -1 & 0 & 0 & -\alpha & 0 & 0 & 0 & -1 & 5 & -1 & 0 \\
0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 5 & -1 \\
0 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & 0 & -1 & 5
\end{pmatrix}. \tag{36}
$$

The determinant can be calculated to be

$$
\begin{aligned}
D_{16(b)}(\theta_1,\theta_2) = 16\{&337746 - 109263(\cos\theta_1+\cos\theta_2) - 109262\cos(\theta_1-\theta_2) \\
&- 4024\cos\theta_1\cos\theta_2 - 4023(\cos\theta_1+\cos\theta_2)\cos(\theta_1-\theta_2) \\
&+ (\cos^3\theta_1+\cos^3\theta_2) + 2118\cos\theta_1\cos\theta_2\cos(\theta_1-\theta_2) \\
&- \sin\theta_1\sin\theta_2(\cos^2\theta_1+\cos^2\theta_2) \\
&- 3\cos\theta_1\cos\theta_2\cos(\theta_1-\theta_2)(\cos\theta_1+\cos\theta_2) \\
&- 2\cos^2\theta_1\cos^2\theta_2\cos(\theta_1-\theta_2)\},
\end{aligned}
\tag{37}
$$

such that

$$
z_{16(b)} = \frac{1}{12}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\ln[D_{16(b)}(\theta_1,\theta_2)] = 1.277\,617\,926\,708\,331\ldots, \tag{38}
$$

which is slightly less than $z_{16(a)}$.

3.1.6. Net 17. Net 17 is obtained by the intergrowth of $3^2.4.3.4$ and 3.4.6.4 vertices. A primitive unit cell containing 28 vertices $\nu_{17} = 28$ is shown in figure 1(*g*). There are sixteen $3^2.4.3.4$ and twelve 3.4.6.4 vertices in each unit cell, so that the effective coordination number is $\kappa_{17} = 32/7$. The matrix $M_{17}(\theta_1, \theta_2)$ is large and omitted here. It is available in the electronic version of this paper in the archive at http://arxiv.org. The determinant can be calculated to be

$$\begin{aligned}
D_{17}(\theta_1, \theta_2) = 16\{&466\,754\,878\,482\,464 - 193\,533\,953\,205\,944(\cos\theta_1 + \cos\theta_2) \\
&+ 341\,192\,376\,4300(\cos^2\theta_1 + \cos^2\theta_2) - 841\,538\,245\,369\,36\cos\theta_1\cos\theta_2 \\
&- 168\,333\,9412(\cos^3\theta_1 + \cos^3\theta_2) \\
&- 118\,550\,064\,1948\cos\theta_1\cos\theta_2(\cos\theta_1 + \cos\theta_2) \\
&+ 388\,09(\cos^4\theta_1 + \cos^4\theta_2) - 729\,383\,812\cos\theta_1\cos\theta_2(\cos^2\theta_1 + \cos^2\theta_2) \\
&+ 188\,549\,748\,86\cos^2\theta_1\cos^2\theta_2 - 110\,32\cos\theta_1\cos\theta_2(\cos^3\theta_1 + \cos^3\theta_2) \\
&- 116\,811\,68\cos^2\theta_1\cos^2\theta_2(\cos\theta_1 + \cos\theta_2) \\
&+ 784\cos^2\theta_1\cos^2\theta_2(\cos^2\theta_1 + \cos^2\theta_2) - 1568\cos^3\theta_1\cos^3\theta_2\},
\end{aligned} \tag{39}$$

such that

$$z_{17} = \frac{1}{28}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\ln[D_{17}(\theta_1, \theta_2)] = 1.299\,177\,753\,544\,099\ldots. \tag{40}$$

3.2. Nets with pentagons

In this subsection, we consider important nets which involve pentagons. Most of them also contain triangles and squares, in addition to pentagons.

3.2.1. Net 18. Net 18 is a pentagon-only net, and is the dual of the $(3^2.4.3.4)$ lattice. The normal appearance that it contains equal-sided (but not regular) pentagons with two angles of $\pi/2$ is not crucial for the calculation of spanning trees. Let us draw it as shown in figure 2(*a*) with six vertices in a unit cell $\nu_{18} = 6$. There are two 5^4 and four 5^3 vertices in each unit cell, so that the effective coordination number is $\kappa_{18} = 10/3$. We have

$$M_{18}(\theta_1, \theta_2) = \begin{pmatrix} 3 & -1 & 0 & 0 & -e^{i(\theta_1+\theta_2)} & -e^{i\theta_2} \\ -1 & 4 & -1 & -e^{i\theta_1} & 0 & -1 \\ 0 & -1 & 3 & -1 & -e^{i\theta_2} & 0 \\ 0 & -e^{-i\theta_1} & -1 & 3 & -1 & 0 \\ -e^{-i(\theta_1+\theta_2)} & 0 & -e^{-i\theta_2} & -1 & 4 & -1 \\ -e^{-i\theta_2} & -1 & 0 & 0 & -1 & 3 \end{pmatrix}. \tag{41}$$

The determinant can be calculated to be

$$D_{18}(\theta_1, \theta_2) = 4\{84 - 36(\cos\theta_1 + \cos\theta_2) + (\cos^2\theta_1 + \cos^2\theta_2) - 14\cos\theta_1\cos\theta_2\}, \tag{42}$$

which is the same as that for the $(3^2.4.3.4)$ lattice as expected. According to equation (5) for $k = 5$, we get

$$\begin{aligned}
z_{18} &= \frac{2}{3}z_{(3^2.4.3.4)} \\
&= \frac{1}{6}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\ln[D_{18}(\theta_1, \theta_2)] = 0.940\,570\,430\,496\,2232\ldots,
\end{aligned} \tag{43}$$

where $z_{(3^2.4.3.4)}$ is given in [18]. A closed-form expression for the integral in equation (43) is given in [25] such that

$$\begin{aligned}
z_{18} = \frac{\ln 6}{3} + \frac{4C}{3\pi} + \frac{8}{3\pi}\Bigg[&\mathrm{Ti}_2(pq, q) + \mathrm{Ti}_2(pq, -q) + \mathrm{Ti}_2\left(\frac{p}{q}, \frac{1}{q}\right) + \mathrm{Ti}_2\left(\frac{p}{q}, -\frac{1}{q}\right) \\
&- \mathrm{Ti}_2\left(pq, \frac{q}{p}\right) - \mathrm{Ti}_2\left(pq, -\frac{q}{p}\right) - \mathrm{Ti}_2\left(\frac{p}{q}, \frac{1}{pq}\right) - \mathrm{Ti}_2\left(\frac{p}{q}, -\frac{1}{pq}\right)\Bigg],
\end{aligned} \tag{44}$$

where $p = \sqrt{3} - \sqrt{2}$, $q = \sqrt{2} - 1$, and $\mathrm{Ti}_2(x, y)$ is the generalized inverse tangent integral [24],

$$\mathrm{Ti}_2(x, y) = \int_0^x \frac{\tan^{-1} t}{t + y}\, \mathrm{d}t. \tag{45}$$

3.2.2. Net 19. Net 19 is shown in figure 2(*b*), where a primitive unit cell contains nine vertices $\nu_{19} = 9$, and the coordination number is $\kappa_{19} = 4$. By the vertex labeling given in figure 2(*b*), we have

$$M_{19}(\theta_1, \theta_2) = \begin{pmatrix} 4 & -1 & -\beta & 0 & 0 & 0 & 0 & -\alpha & -1 \\ -1 & 4 & -1 & -1 & 0 & -\alpha & 0 & 0 & 0 \\ -\frac{1}{\beta} & -1 & 4 & -1 & 0 & -\alpha & 0 & 0 & 0 \\ 0 & -1 & -1 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 4 & -1 & 0 & -\frac{1}{\beta} & -\frac{1}{\beta} \\ 0 & -\frac{1}{\alpha} & -\frac{1}{\alpha} & 0 & -1 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 & -1 \\ -\frac{1}{\alpha} & 0 & 0 & 0 & -\beta & 0 & -1 & 4 & -1 \\ -1 & 0 & 0 & 0 & -\beta & 0 & -1 & -1 & 4 \end{pmatrix}. \tag{46}$$

The determinant can be calculated to be

$$\begin{aligned} D_{19}(\theta_1, \theta_2) = 80\{&442 - 191(\cos\theta_1 + \cos\theta_2) + 5(\cos^2\theta_1 + \cos^2\theta_2) \\ &- 68\cos\theta_1\cos\theta_2 - \cos\theta_1\cos\theta_2(\cos\theta_1 + \cos\theta_2)\}, \end{aligned} \tag{47}$$

such that

$$z_{19} = \frac{1}{9}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_2}{2\pi}\ln[D_{19}(\theta_1, \theta_2)] = 1.144\,188\,002\,944\,693\ldots. \tag{48}$$

3.2.3. Net 20. Net 20 is shown in figure 2(*c*), where a primitive unit cell contains ten vertices $\nu_{20} = 10$, and the coordination number is $\kappa_{20} = 4$. We have

$$M_{20}(\theta_1, \theta_2) = \begin{pmatrix} 4 & -1 & 0 & -\beta & 0 & 0 & -\alpha\beta & 0 & -\alpha & 0 \\ -1 & 4 & -1 & 0 & 0 & -\beta & 0 & 0 & 0 & -1 \\ 0 & -1 & 4 & -1 & -1 & 0 & 0 & 0 & -\alpha & 0 \\ -\frac{1}{\beta} & 0 & -1 & 4 & -1 & 0 & -\alpha & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -\frac{1}{\beta} & 0 & 0 & -1 & 4 & -1 & 0 & 0 & -\frac{1}{\beta} \\ -\frac{1}{\alpha\beta} & 0 & 0 & -\frac{1}{\alpha} & 0 & -1 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 & -1 \\ -\frac{1}{\alpha} & 0 & -\frac{1}{\alpha} & 0 & 0 & 0 & 0 & -1 & 4 & -1 \\ 0 & -1 & 0 & 0 & 0 & -\beta & 0 & -1 & -1 & 4 \end{pmatrix}. \tag{49}$$

The determinant can be calculated to be

$$\begin{aligned} D_{20}(\theta_1, \theta_2) = 16\{&7375 - 2995(\cos\theta_1 + \cos\theta_2) + 34(\cos^2\theta_1 + \cos^2\theta_2) \\ &- 1393\cos\theta_1\cos\theta_2 - 30\cos\theta_1\cos\theta_2(\cos\theta_1 + \cos\theta_2)\}, \end{aligned} \tag{50}$$

such that

$$z_{20} = \frac{1}{10}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_2}{2\pi}\ln[D_{20}(\theta_1, \theta_2)] = 1.150\,677\,474\,300\,389\ldots. \tag{51}$$

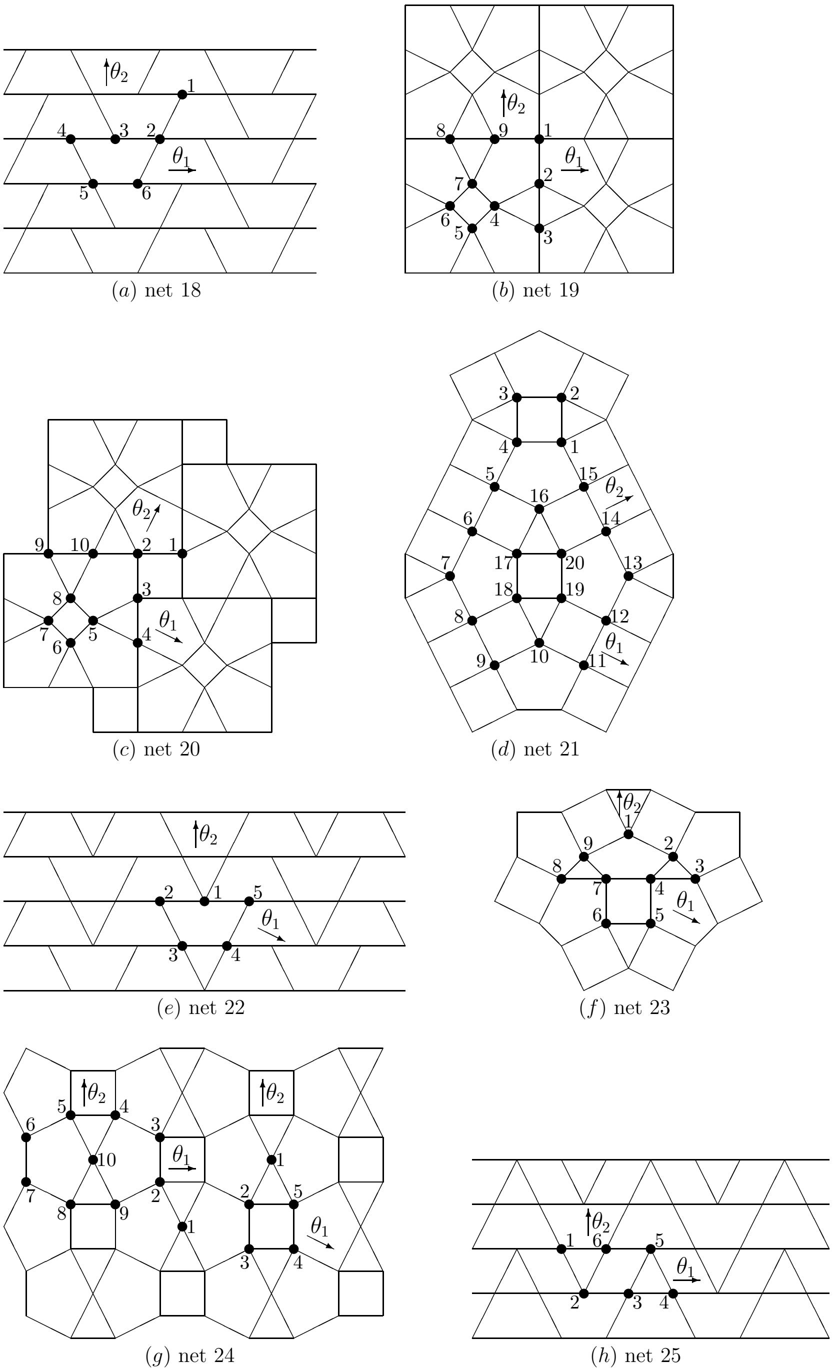


Figure 2. Unit cells for nets 18–25. Vertices within a unit cell are labeled. Directions θ_1 and θ_2 are sketched. There are two kinds of unit cells for net 24.

3.2.4. Net 21. A primitive unit cell for net 21 containing 20 vertices $\nu_{21} = 20$ is shown in figure 2(*d*), and the coordination number is $\kappa_{21} = 4$. The matrix $M_{21}(\theta_1, \theta_2)$ is large and omitted here. It is available in the electronic version of this paper in the archive at http://arxiv.org. The determinant can be calculated to be

$$\begin{aligned} D_{21}(\theta_1,\theta_2) = 16\{&816\,275\,712 - 342\,967\,936(\cos\theta_1+\cos\theta_2) + 676\,6760(\cos^2\theta_1+\cos^2\theta_2) \\ &- 139\,108\,816\cos\theta_1\cos\theta_2 - 7200(\cos^3\theta_1+\cos^3\theta_2) \\ &- 238\,9920\cos\theta_1\cos\theta_2(\cos\theta_1+\cos\theta_2) + (\cos^4\theta_1+\cos^4\theta_2) \\ &- 1160\cos\theta_1\cos\theta_2(\cos^2\theta_1+\cos^2\theta_2) + 320\,14\cos^2\theta_1\cos^2\theta_2\}, \end{aligned} \tag{52}$$

such that

$$z_{21} = \frac{1}{20}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\ln[D_{21}(\theta_1,\theta_2)] = 1.155\,959\,257\,782\,222\ldots. \tag{53}$$

3.2.5. Net 22. Net 22 contains only triangles and pentagons. Let us draw it as shown in figure 2(*e*) with five vertices in a unit cell $\nu_{22} = 5$. There are two 5^3 and three $5^3.3$ vertices in each unit cell, so that the effective coordination number is $\kappa_{22} = 18/5$. We have

$$M_{22}(\theta_1,\theta_2) = \begin{pmatrix} 4 & -1 & -e^{i\theta_2} & -e^{i\theta_2} & -1 \\ -1 & 3 & -1 & -e^{-i\theta_1} & 0 \\ -e^{-i\theta_2} & -1 & 4 & -1 & -e^{-i(\theta_1+\theta_2)} \\ -e^{-i\theta_2} & -e^{i\theta_1} & -1 & 4 & -1 \\ -1 & 0 & -e^{i(\theta_1+\theta_2)} & -1 & 3 \end{pmatrix}. \tag{54}$$

The determinant can be calculated to be

$$\begin{aligned} D_{22}(\theta_1,\theta_2) = 2\{&99 - 31(\cos\theta_1+\cos\theta_2+\cos(\theta_1+\theta_2)) \\ &-2(\cos\theta_1\cos\theta_2 + (\cos\theta_1+\cos\theta_2)\cos(\theta_1+\theta_2))\}, \end{aligned} \tag{55}$$

such that

$$z_{22} = \frac{1}{5}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\ln[D_{22}(\theta_1,\theta_2)] = 1.024\,172\,110\,372\,259\ldots. \tag{56}$$

3.2.6. Net 23. A primitive unit cell for net 23 containing nine vertices $\nu_{23} = 9$ is shown in figure 2(*f*), and the coordination number is $\kappa_{23} = 4$. We have

$$M_{23}(\theta_1,\theta_2) = \begin{pmatrix} 4 & -1 & 0 & 0 & -\beta & -\beta & 0 & 0 & -1 \\ -1 & 4 & -1 & -1 & 0 & 0 & 0 & -\alpha\beta & 0 \\ 0 & -1 & 4 & -1 & 0 & -\alpha\beta & 0 & 0 & -\alpha \\ 0 & -1 & -1 & 4 & -1 & 0 & -1 & 0 & 0 \\ -\frac{1}{\beta} & 0 & 0 & -1 & 4 & -1 & 0 & -\alpha & 0 \\ -\frac{1}{\beta} & 0 & -\frac{1}{\alpha\beta} & 0 & -1 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 & -1 \\ 0 & -\frac{1}{\alpha\beta} & 0 & 0 & -\frac{1}{\alpha} & 0 & -1 & 4 & -1 \\ -1 & 0 & -\frac{1}{\alpha} & 0 & 0 & 0 & -1 & -1 & 4 \end{pmatrix}. \tag{57}$$

The determinant can be calculated to be

$$\begin{aligned} D_{23}(\theta_1,\theta_2) = 8\{&4714 - 1473(\cos\theta_1+\cos\theta_2+\cos(\theta_1+\theta_2)) - 103(\cos\theta_1\cos\theta_2 \\ &+ (\cos\theta_1+\cos\theta_2)\cos(\theta_1+\theta_2)) + 14\cos\theta_1\cos\theta_2\cos(\theta_1+\theta_2)\}, \end{aligned} \tag{58}$$

such that

$$z_{23} = \frac{1}{9}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\ln[D_{23}(\theta_1,\theta_2)] = 1.152\,329\,841\,150\,063\ldots. \tag{59}$$

3.2.7. Net 24. The coordination number for net 24 is $\kappa_{24} = 4$. The unit cell given in figure 26 of [1] contains ten vertices $\nu_{24} = 10$. Using the vertex labeling given in the left-hand side of figure 2(*g*), we have

$$M_{24}(\theta_1,\theta_2) = \begin{pmatrix} 4 & -1 & -\frac{1}{\beta} & 0 & 0 & -\frac{\alpha}{\beta} & -\alpha & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 & 0 & -\alpha & 0 & -1 & 0 \\ -\beta & -1 & 4 & -1 & 0 & -\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & -1 & 0 & 0 & 0 & -\beta & -1 \\ 0 & 0 & 0 & -1 & 4 & -1 & 0 & -\beta & 0 & -1 \\ -\frac{\beta}{\alpha} & 0 & -\frac{1}{\alpha} & 0 & -1 & 4 & -1 & 0 & 0 & 0 \\ -\frac{1}{\alpha} & -\frac{1}{\alpha} & 0 & 0 & 0 & -1 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{\beta} & 0 & -1 & 4 & -1 & -1 \\ 0 & -1 & 0 & -\frac{1}{\beta} & 0 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & -1 & 4 \end{pmatrix}, \tag{60}$$

with determinant

$$\begin{aligned} D_{24}(\theta_1,\theta_2) &= 4\{30135 - 5929\cos\theta_1 - 23101\cos\theta_2 + 36\cos^2\theta_1 + 2456\cos^2\theta_2 \\ &\quad - 3341\cos\theta_1\cos\theta_2 - 4\cos^3\theta_2 - 248\cos\theta_1\cos^2\theta_2 - 4\cos\theta_1\cos^3\theta_2\} \\ &= 16\left(121 - 6\cos^2\frac{\theta_1}{2} - 52\cos^2\frac{\theta_2}{2} - 59\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - 4\cos\frac{\theta_1}{2}\cos^3\frac{\theta_2}{2}\right) \\ &\quad\times\left(121 - 6\cos^2\frac{\theta_1}{2} - 52\cos^2\frac{\theta_2}{2} + 59\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + 4\cos\frac{\theta_1}{2}\cos^3\frac{\theta_2}{2}\right). \end{aligned} \tag{61}$$

However, if one takes the primitive unit cell shown in the right-hand side of figure 2(*g*) containing five vertices, $\bar{\nu}_{24} = 5$, we have

$$\bar{M}_{24}(\theta_1,\theta_2) = \begin{pmatrix} 4 & -1 & -e^{i\theta_2} & -e^{i\theta_2} & -1 \\ -1 & 4 & -1 & -e^{-i\theta_1} & -1 \\ -e^{-i\theta_2} & -1 & 4 & -1 & -e^{-i(\theta_1+\theta_2)} \\ -e^{-i\theta_2} & -e^{i\theta_1} & -1 & 4 & -1 \\ -1 & -1 & -e^{i(\theta_1+\theta_2)} & -1 & 4 \end{pmatrix}, \tag{62}$$

with determinant

$$\begin{aligned} \bar{D}_{24}(\theta_1,\theta_2) &= 2\{184 - 61\cos\theta_1 - 46\cos\theta_2 - 61\cos(\theta_1+\theta_2) - 2\cos\theta_1\cos\theta_2 \\ &\quad -12\cos\theta_1\cos(\theta_1+\theta_2) - 2\cos\theta_2\cos(\theta_1+\theta_2)\}. \end{aligned} \tag{63}$$

Although the two determinants given in equations (61) and (63) are distinct, they give the same entropy for net 24,

$$\begin{aligned} z_{24} &= \frac{1}{10}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\ln[D_{24}(\theta_1,\theta_2)] \\ &= \frac{1}{5}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\ln[\bar{D}_{24}(\theta_1,\theta_2)] = 1.148\,658\,687\,301\,195\ldots. \end{aligned} \tag{64}$$

This is an example of the integral identity one can obtain by choosing different unit cells in the calculation [18].

3.2.8. Net 25. Let us draw net 25 as shown in figure 2(*h*) with six vertices in a unit cell $\nu_{25} = 6$, and the coordination number is $\kappa_{25} = 4$. We have

$$M_{25}(\theta_1, \theta_2) = \begin{pmatrix} 4 & -1 & 0 & -\mathrm{e}^{\mathrm{i}(\theta_2-\theta_1)} & -\mathrm{e}^{-\mathrm{i}\theta_1} & -1 \\ -1 & 4 & -1 & -\mathrm{e}^{-\mathrm{i}\theta_1} & 0 & -1 \\ 0 & -1 & 4 & -1 & -1 & -\mathrm{e}^{-\mathrm{i}\theta_2} \\ -\mathrm{e}^{\mathrm{i}(\theta_1-\theta_2)} & -\mathrm{e}^{\mathrm{i}\theta_1} & -1 & 4 & -1 & 0 \\ -\mathrm{e}^{\mathrm{i}\theta_1} & 0 & -1 & -1 & 4 & -1 \\ -1 & -1 & -\mathrm{e}^{\mathrm{i}\theta_2} & 0 & -1 & 4 \end{pmatrix}. \tag{65}$$

The determinant can be calculated to be

$$\begin{aligned} D_{25}(\theta_1, \theta_2) = 4\{&295 - 104\cos\theta_1 - 136\cos\theta_2 + \cos^2\theta_1 + 2\cos^2\theta_2 \\ &- 56\cos\theta_1\cos\theta_2 - 2\cos\theta_1\cos^2\theta_2\}, \end{aligned} \tag{66}$$

such that

$$z_{25} = \frac{1}{6}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_2}{2\pi}\ln[D_{25}(\theta_1, \theta_2)] = 1.150\,037\,457\,106\,072\ldots. \tag{67}$$

3.3. Nets with heptagons, enneagons or octagons

In this subsection, we consider important nets which involve heptagons, enneagons or octagons.

3.3.1. Net 26. Net 26 contains only squares and heptagons. A primitive unit cell containing 12 vertices $\nu_{26} = 12$ is shown in figure 3(*a*), and the coordination number is $\kappa_{26} = 3$. We have

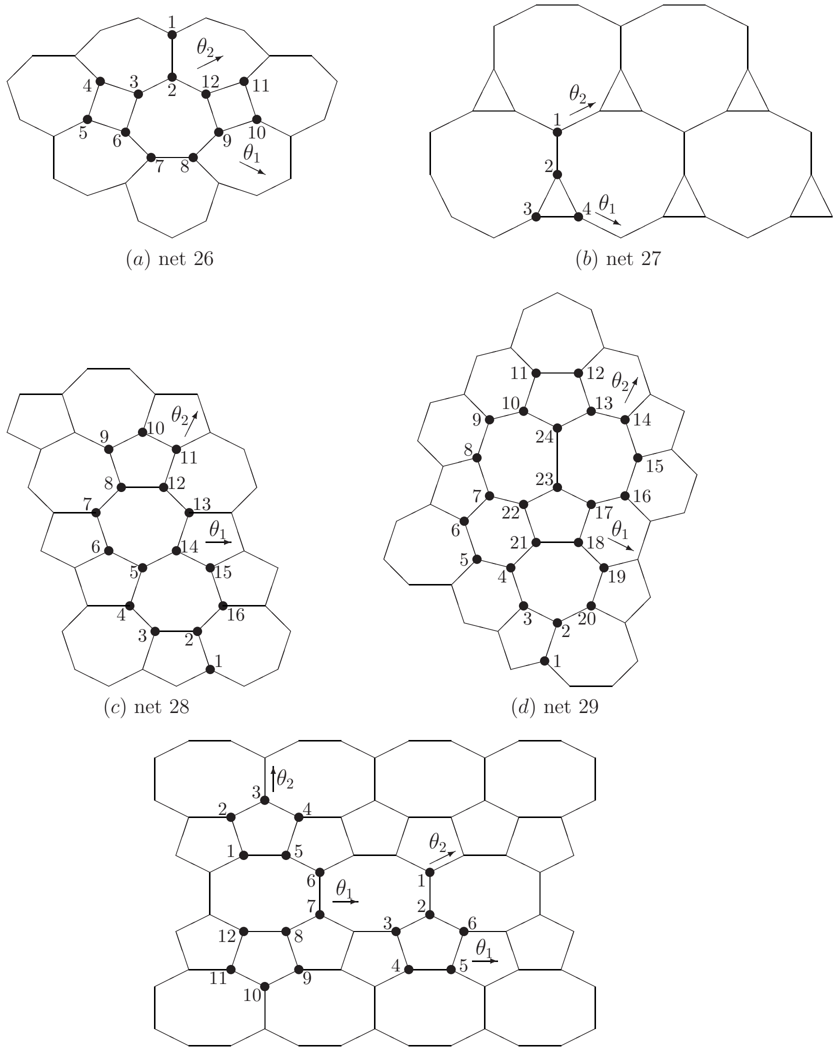
$$M_{26}(\theta_1, \theta_2) = \begin{pmatrix} 3 & -1 & 0 & 0 & -\beta & 0 & 0 & 0 & 0 & -\frac{1}{\alpha} & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 3 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & 0 & 0 & -\frac{1}{\alpha} & 0 & 0 & 0 & 0 \\ -\frac{1}{\beta} & 0 & 0 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 & 0 & 0 & -\frac{1}{\beta} & 0 \\ 0 & 0 & 0 & -\alpha & 0 & 0 & -1 & 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 & 0 & -1 \\ -\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\beta & 0 & 0 & -1 & 3 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 3 \end{pmatrix}. \tag{68}$$

The determinant can be calculated to be

$$\begin{aligned} D_{26}(\theta_1, \theta_2) = 4\{&4140 - 1752(\cos\theta_1 + \cos\theta_2) + 37(\cos^2\theta_1 + \cos^2\theta_2) \\ &- 686\cos\theta_1\cos\theta_2 - 12\cos\theta_1\cos\theta_2(\cos\theta_1 + \cos\theta_2)\}, \end{aligned} \tag{69}$$

such that

$$z_{26} = \frac{1}{12}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta_2}{2\pi}\ln[D_{26}(\theta_1, \theta_2)] = 0.795\,040\,242\,875\,7831\ldots. \tag{70}$$



(a) net 26 (b) net 27 (c) net 28 (d) net 29 (e) net 30

Figure 3. Unit cells for nets 26–30. Vertices within a unit cell are labeled. Directions θ_1 and θ_2 are sketched. There are two kinds of unit cells for net 30.

3.3.2. Net 27. Net 27 contains only triangles and enneagons as shown in figure 3(*b*), where each unit cell contains four vertices $\nu_{27} = 4$, and the coordination number is $\kappa_{27} = 3$. We have

$$M_{27}(\theta_1, \theta_2) = \begin{pmatrix} 3 & -1 & -e^{i\theta_2} & -e^{-i\theta_1} \\ -1 & 3 & -1 & -1 \\ -e^{-i\theta_2} & -1 & 3 & -1 \\ -e^{i\theta_1} & -1 & -1 & 3 \end{pmatrix}. \tag{71}$$

The determinant can be calculated to be

$$D_{27}(\theta_1, \theta_2) = 8\{3 - (\cos\theta_1 + \cos\theta_2 + \cos(\theta_1 + \theta_2))\}, \tag{72}$$

which is four times of that for the triangular lattice [8, 16, 26]. It follows that

$$z_{27} = \frac{1}{4}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\ln[D_{27}(\theta_1,\theta_2)] = \frac{1}{4}\left(z_{\rm tri} + \ln 4\right)$$
$$= \frac{3\sqrt{3}}{4\pi}\left(1 - \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{11^2} + \frac{1}{13^2} - \cdots\right) + \frac{\ln 4}{4}$$
$$= 0.750\,406\,024\,304\,2857\ldots. \tag{73}$$

This expression of z_{27} in terms of $z_{\rm tri}$ is similar to those for the Kagomé and (3.12.12) lattices [16].

3.3.3. Net 28. Net 28 is the B net of $YCrB_4$, which contains only pentagons and heptagons. A primitive unit cell containing 16 vertices $\nu_{28} = 16$ is shown in figure 3(*c*), and the coordination number is $\kappa_{28} = 3$. The matrix $M_{28}(\theta_1,\theta_2)$ is large and omitted here. It is available in the electronic version of this paper in the archive at http://arxiv.org. The determinant can be calculated to be

$$D_{28}(\theta_1,\theta_2) = 4\{121\,795 - 115\,860\cos\theta_1 - 114\,28\cos\theta_2 + 197\,77\cos^2\theta_1 + 2\cos^2\theta_2$$
$$- 114\,72\cos\theta_1\cos\theta_2 - 628\cos^3\theta_1 - 2132\cos^2\theta_1\cos\theta_2$$
$$- 2\cos\theta_1\cos^2\theta_2 + 4\cos^4\theta_1 - 56\cos^3\theta_1\cos\theta_2\}, \tag{74}$$

such that

$$z_{28} = \frac{1}{16}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\ln[D_{28}(\theta_1,\theta_2)] = 0.802\,588\,172\,310\,6822\ldots. \tag{75}$$

3.3.4. Net 29. Net 29 is the B net of $Y_2\,LnB_6$, which contains pentagons, hexagons and heptagons. A primitive unit cell containing 24 vertices $\nu_{29} = 24$ is shown in figure 3(*d*), and the coordination number is $\kappa_{29} = 3$. The matrix $M_{29}(\theta_1,\theta_2)$ is large and omitted here. It is available in the electronic version of this paper in the archive at http://arxiv.org. The determinant can be calculated to be

$$D_{29}(\theta_1,\theta_2) = 8\{364\,077\,67 - 213\,924\,74\cos\theta_1 - 991\,3039\cos\theta_2 + 103\,5241\cos^2\theta_1$$
$$+ 476\,42\cos^2\theta_2 - 586\,1590\cos\theta_1\cos\theta_2 - 4068\cos^3\theta_1$$
$$- 288\,591\cos^2\theta_1\cos\theta_2 - 311\,98\cos\theta_1\cos^2\theta_2 - 2\cos^3\theta_2 + 2\cos^4\theta_1$$
$$- 840\cos^3\theta_1\cos\theta_2 + 1156\cos^2\theta_1\cos^2\theta_2 - 2\cos\theta_1\cos^3\theta_2$$
$$- 4\cos^3\theta_1\cos^2\theta_2\}, \tag{76}$$

such that

$$z_{29} = \frac{1}{24}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\ln[D_{29}(\theta_1,\theta_2)] = 0.804\,388\,017\,977\,0491\ldots. \tag{77}$$

3.3.5. Net 30. Net 30 is a net with $5^2.8$ and 5.8^2 vertices. Its coordination number is $\kappa_{30} = 3$. The unit cell given in figure 31 of [1] contains 12 vertices $\nu_{30} = 12$. Using the vertex labeling given in the left-hand side of figure 3(*e*), we have

$$M_{30}(\theta_1,\theta_2) = \begin{pmatrix} 3 & -1 & 0 & 0 & -1 & -\frac{1}{\alpha} & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & -\frac{1}{\alpha} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & -\beta & 0 & 0 \\ 0 & -\alpha & -1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\alpha & 0 & 0 & 0 & -1 & 3 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 & 0 & 0 & 0 & -\alpha \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 & -\alpha & 0 \\ 0 & 0 & -\frac{1}{\beta} & 0 & 0 & 0 & 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\alpha} & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\alpha} & -1 & 0 & 0 & -1 & 3 \end{pmatrix}, \tag{78}$$

with determinant

$$\begin{aligned} D_{30}(\theta_1,\theta_2) &= 4\{5160 - 6791\cos\theta_1 - 169\cos\theta_2 + 2261\cos^2\theta_1 - 221\cos\theta_1\cos\theta_2 \\ &\quad - 184\cos^3\theta_1 - 56\cos^2\theta_1\cos\theta_2 + 4\cos^4\theta_1 - 4\cos^3\theta_1\cos\theta_2\} \\ &= 16\left(60 - 49\cos^2\frac{\theta_1}{2} + 11\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + 4\cos^4\frac{\theta_1}{2} + 4\cos^3\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\right) \\ &\quad \times \left(60 - 49\cos^2\frac{\theta_1}{2} - 11\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + 4\cos^4\frac{\theta_1}{2} - 4\cos^3\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\right). \end{aligned} \tag{79}$$

However, if one takes the primitive unit cell shown in the right-hand side of figure 3(*e*), where each unit cell contains six vertices, $\bar{\nu}_{30} = 6$, we have

$$\bar{M}_{30}(\theta_1,\theta_2) = \begin{pmatrix} 3 & -1 & 0 & -e^{i\theta_2} & -e^{i(\theta_2-\theta_1)} & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 \\ 0 & -1 & 3 & -1 & 0 & -e^{-i\theta_1} \\ -e^{-i\theta_2} & 0 & -1 & 3 & -1 & 0 \\ -e^{i(\theta_1-\theta_2)} & 0 & 0 & -1 & 3 & -1 \\ 0 & -1 & -e^{i\theta_1} & 0 & -1 & 3 \end{pmatrix}, \tag{80}$$

with determinant

$$\begin{aligned} \bar{D}_{30}(\theta_1,\theta_2) &= 2\{73 - 45\cos\theta_1 - 13\cos\theta_2 - 13\cos(\theta_1-\theta_2) + 2\cos^2\theta_1 \\ &\quad - 2\cos\theta_1\cos\theta_2 - 2\cos\theta_1\cos(\theta_1-\theta_2)\}. \end{aligned} \tag{81}$$

Although the two determinants given in equations (79) and (81) are distinct, they give the same entropy for net 30,

$$\begin{aligned} z_{30} &= \frac{1}{12}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\ln[D_{30}(\theta_1,\theta_2)] \\ &= \frac{1}{6}\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\ln[\bar{D}_{30}(\theta_1,\theta_2)] = 0.798\,501\,354\,579\,1521\ldots. \end{aligned} \tag{82}$$

This is another example of the integral identity one can obtain by choosing different unit cells in the calculation [18].

Table 1. Number of vertices in a primitive unit cell ν_Λ, effective coordination number κ_Λ and numerical values of z_Λ and r_Λ. The last digits given in the text are rounded off.

Λ	ν_Λ	κ_Λ	z_Λ	r_Λ
net 12	12	5	1.409 737 903 756 93	0.950 533 252 598 831
net 13	8	5	1.409 133 286 424 68	0.950 125 581 869 492
net 14	3	4	1.127 778 363 805 54	0.927 147 894 482 279
net 15	11	$\frac{42}{11}$	1.073 270 254 423 06	0.926 486 463 318 153
net 16(a)	12	$\frac{9}{2}$	1.280 287 248 642 48	0.941 934 677 506 853
net 16(b)	12	$\frac{9}{2}$	1.277 617 926 708 33	0.939 970 800 339 543
net 17	28	$\frac{32}{7}$	1.299 177 753 544 10	0.942 842 965 374 234
net 18	6	$\frac{10}{3}$	0.940 570 430 496 223	0.957 234 196 587 022
net 19	9	4	1.144 188 002 944 69	0.940 638 277 757 358
net 20	10	4	1.150 677 474 300 39	0.945 973 279 648 539
net 21	20	4	1.155 959 257 782 22	0.950 315 439 944 796
net 22	5	$\frac{18}{5}$	1.024 172 110 372 26	0.945 174 625 646 314
net 23	9	4	1.152 329 841 150 06	0.947 331 692 342 695
net 24	5	4	1.148 658 687 301 20	0.944 313 632 526 569
net 25	6	4	1.150 037 457 106 07	0.945 447 120 774 431
net 26	12	3	0.795 040 242 875 783	0.949 882 240 790 510
net 27	4	3	0.750 406 024 304 286	0.896 555 064 043 746
net 28	16	3	0.802 588 172 310 682	0.958 900 204 584 415
net 29	24	3	0.804 388 017 977 049	0.961 050 588 102 690
net 30	6	3	0.798 501 354 579 152	0.954 017 438 436 016

4. Discussion

It is of interest to see how close the exact results presented above are to the upper bound given in equation (8). For this purpose, we define the ratio

$$r_{\Lambda_k} = \frac{z_{\Lambda_k}}{\ln b_k} \tag{83}$$

for a k-regular lattice Λ_k, where b_k is given by equation (7). For a lattice Λ which is not k-regular, we replace k by κ in equation (7) and consider the ratio

$$r_\Lambda = \frac{z_\Lambda}{\ln b_\kappa}. \tag{84}$$

The values of z_Λ and r_Λ for various lattices Λ are summarized in table 1. Our results agree with the observations made in [16] that z_Λ is relatively large for the large value of k (or κ). For the two-dimensional lattices with $k = 4$ studied here and in [16, 18], their values of z_Λ are all smaller than that of the square lattice, $z_{sq} = 4C/\pi = 1.166\,243\,616\,123\,275\ldots$ [8], which indicates that the square lattice may be the most densely connected two-dimensional lattice with $k = 4$ [16]. Similarly, the values of z_Λ for the lattices with $k = 3$ studied here and in [16, 18] are smaller than that of the honeycomb lattice, $z_{\rm hc} = 0.807\,664\,868\,048\,6262\ldots$ [8], which indicates that the honeycomb lattice may be the most densely connected two-dimensional lattice with $k = 3$. The triangular lattice, the dual of the honeycomb lattice, would be the most densely connected two-dimensional lattice with $k = 6$.

The ratios r_Λ are close to each other no matter whether the lattice Λ is k-regular or not. We thereby conjecture the upper bound

$$z_\Lambda \leqslant \ln(b_\kappa) \tag{85}$$

for a lattice Λ with an effective coordination number $\kappa \geqslant 3$, which generalizes the bound given in equation (8) for k-regular lattices.

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